REAL ESTATE THEORY AND MODELLING

INSTITUT DE L'ÉPARGNE IMMOBILIÈRE & FONCIÈRE

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CHRONICLE N°7

The Gordon-Shapiro model for real estate (2/4)

We showed in Chronicle 6 that Gordon-Shapiro's formula allows us to write the total return (tr) as the sum of the initial net income return (ir_1) plus the growth rate of net operating income (g) under the assumptions that the growth rate of net operating income is strictly constant over time and that we have an infinite investment horizon.

$$ir_1 = tr - g \Leftrightarrow tr = ir_1 + g$$

The assumption of strict constancy in the net operating income growth rate

In order to show you the sensitivity of the model to the strict constancy of the growth rate g, I will present the results of a simple model implemented in Excel.

It is based on the general formula of today's price, which is equal to the discounted sum of future revenues (for details, see Chronicle No. 6):

$$(1)\ P_0 = \sum_{t=1}^{\infty} \frac{noi_t}{(1+r)^t}$$

$$(2)\ P_0 = \frac{noi_1}{(1+r)} + \frac{noi_2}{(1+r)^2} + \cdots + \frac{noi_t}{(1+r)^t} + \cdots$$

with:
- P_0 : the price today
- noi_t : the net operating income for period t
- r : the discount rate

Under the assumption:

$$(3)\ noi_t = (1+g)^t . noi_0$$

with:
- g : the growth rate of the net operating income

Applied step by step in Excel, these formulae give the following result for the following assumptions: noi_1=10, g =3% and r =5%.

Table 1

t	noi	g	r	$(1+r)^t$	$noi_t/(1+r)^t$	P_0 (g constant)
1	10.0		5%	1.05	9.52	9.52
2	10.3	3%	5%	1.10	9.34	18.87
3	10.6	3%	5%	1.16	9.16	28.03
4	10.9	3%	5%	1.22	8.99	37.02
5	11.3	3%	5%	1.28	8.82	45.84
6	11.6	3%	5%	1.34	8.65	54.49
7	11.9	3%	5%	1.41	8.49	62.98
8	12.3	3%	5%	1.48	8.32	71.30
9	...	...	...	...	...	...

Example of how to calculate P_0 :
For t=2: P_0 = 9.52 + 9.34 = 18.87 (to the nearest whole number...)
For t=3: P_0 = 9.52 + 9.34 + 9.16 = 18.87 + 9.34 = 28.03 (to the nearest whole number...)

After 100 iterations, for t=100, we find P_0=426.92...
After 250 iterations, for t=250, we find P_0=495.91...

To what exact value does this series converge?

It's very easy to calculate using the Gordon-Shapiro formula.

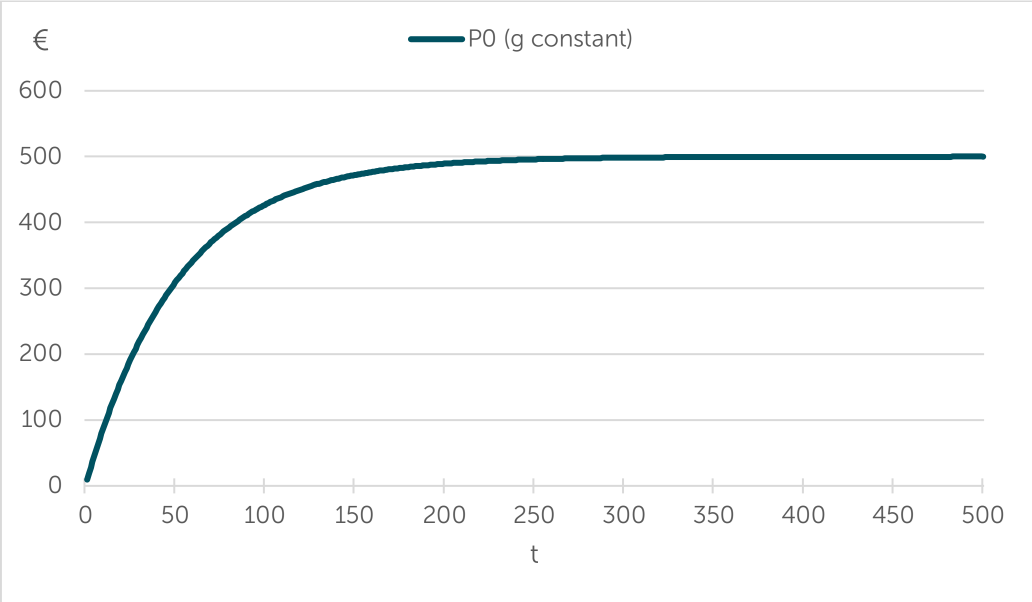
Since Chronicle 6, we know that:

$$ir_1 = tr - g \Leftrightarrow ir_1 = r + g \Leftrightarrow \frac{noi_1}{P_0} = r + g$$

$$\Leftrightarrow \frac{noi_1}{r + g} = P_0$$

Since we know the initial net rental income (noi_1), the discount rate (r) and the strictly constant growth rate of net rental income (g), we know the price (P_0).

$$P_0 = \frac{noi_1}{r + g} = \frac{10}{5\% - 3\%} = 500$$



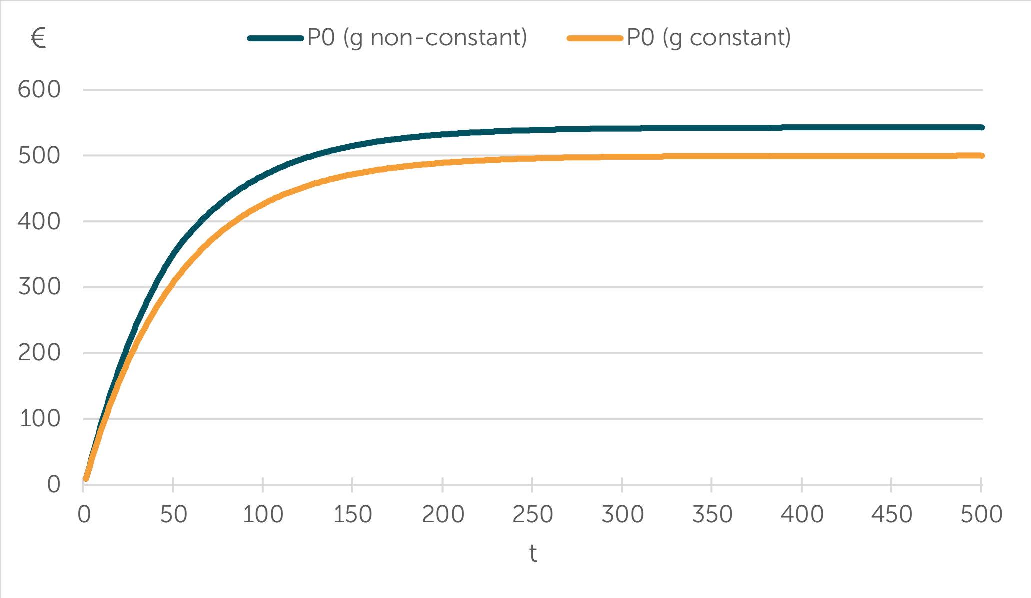
What happens now if I assume that the income growth rate is not strictly constant but that its arithmetic average over the period is the same, i.e. 3%?

To do this, I'm going to change the first 5 growth rates for periods 2 to 6 to 6% and the growth rates for the 5 periods between 50 and 54 to 0%. In this way, the arithmetic mean of the growth rate over the period is maintained, at exactly 3%.

Table 2

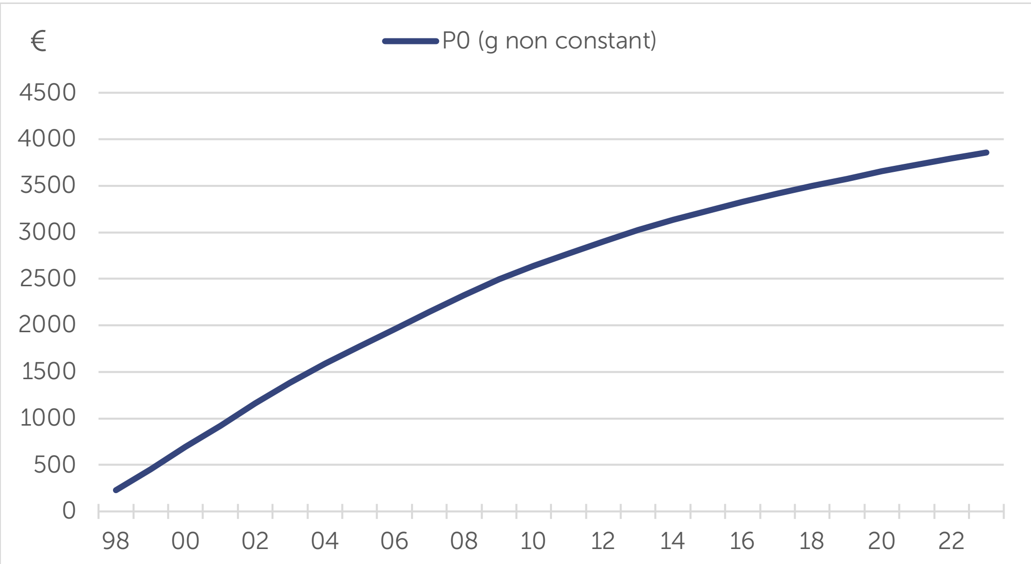
t	noi	g	r	$(1+r)^t$	$noi_t/(1+r)^t$	P_0
1	10.0		5%	1.05	9.52	9.52
2	10.6	6%	5%	1.10	9.61	19.14
3	11.2	6%	5%	1.16	9.71	28.84
4	11.9	6%	5%	1.22	9.80	38.64
5	12.6	6%	5%	1.28	9.89	48.53
6	13.4	6%	5%	1.34	9.99	58.52
7	13.8	3%	5%	1.41	9.80	68.32
8	...	...	...	...	...	...
49	...	...	...	...	...	...
50	47.7	0%	5%	11.47	4.16	352.02
51	47.7	0%	5%	12.04	3.96	355.99
52	47.7	0%	5%	12.64	3.77	359.76
53	47.7	0%	5%	13.27	3.59	363.35
54	47.7	0%	5%	13.94	3.42	366.77
55	49.1	3%	5%	14.64	3.36	370.13
56	...	...	...	...	...	...

Well, as anyone familiar with financial mathematics knows, and as you can see from the graph, the price calculated by the second model converges to a different value from the first, towards 543 and not 500, a difference of nearly 10%...



Furthermore, as regards the dynamics of net operating income, since we are dealing with a geometric rather than an arithmetic sequence, in order to be rigorous we need to use the geometric average rather than the arithmetic average.

Let's take another example, **based on real data from the Paris market, and compare it with the results obtained using the geometric mean.**



source : MSCI, IEIF calculations

Table 3

t	noi/m^2	g	rm	$(1+r)^t$	$noi_t/(1+r)^t$	P_0
Dec 1998	228,16		8,16	1,00	228,16	228,16
Dec 1999	247,38	8,42	8,16	1,08	228,71	456,87
Dec 2000	276,33	11,70	8,16	1,17	236,20	693,08
Dec 2001	292,24	5,76	8,16	1,27	230,95	924,03
Dec 2002	325,94	11,53	8,16	1,37	238,15	1 162,18
…	…	…	…	…	…	…
…	…	…	…	…	…	…
Dec 2019	415,42	7,43	8,16	5,19	79,98	3 576,53
Dec 2020	434,08	4,49	8,16	5,62	77,27	3 653,79
Dec 2021	440,18	1,41	8,16	6,08	72,44	3 726,23
Dec 2022	437,51	-0,61	8,16	6,57	66,57	3 792,80
Dec 2023	458,93	4,90	8,16	7,11	64,56	3 857,36

Source MSCI, IEIF calculation

With:
- noi/m^2 the average net operation income per square metre in the Paris office market according to MSCI data
- g the net operating income growth rate,
- rm is the geometric mean of the total return on the Paris office market according to MSCI data

$$rm = \sqrt[T]{\prod_{t=0}^{T}(1+r_t)} - 1$$

With:
- rm : the geometric mean of the total return
- r_t : the overall return in period t
- T : the total number of periods used to calculate the average

If we return to the price calculation, we can calculate the associated value of P_0 for each iteration. Thus, for the data corresponding to December 2023, P_0 is equal to 3,857.36.

$$P_0 = 3\,857.36 = 228.16 + 228.71 + \ldots + 66.57 + 64.56$$

What happens to this value P_0 if I use, not the actual series of rental income and its associated growth rate (table 3), but those resulting from the application to each period of the geometric mean growth rate (average calculated between December 1998 and December 2023) and its associated rental income (table 4)?

The geometric average of g over the period is 2.83%, i.e. if I increase the initial net rental income (228.16 euros) by 2.83% a year between 1998 and 2023, I find 458.93 at the end of the period, i.e. the net rental income actually recorded for the year 2023.

So I get the same result in my net operating income calculation if I use the average growth rate over the period or if I use the real growth rate series.

But what about the value of P_0?

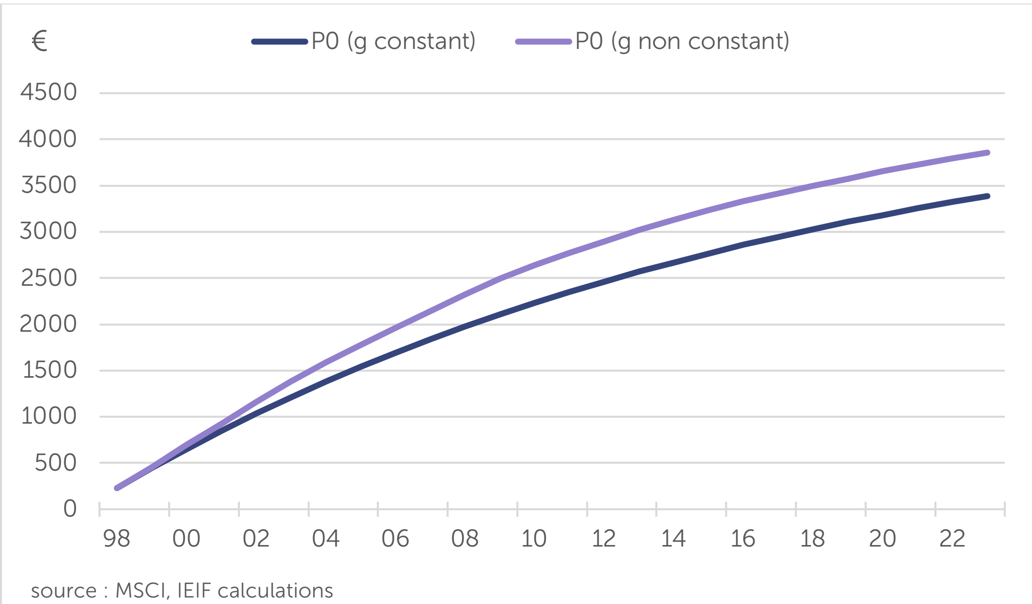
In table 4 I use the geometric mean of the growth rate (2.83%) applied to the initial rental income (228.16 euros), and I calculate P_0.

Table 4

t	noi/m²	g	r	$(1+r)^t$	$noi_t/(1+r)^t$	P_0
Dec 1998	228.16		8.16	1.00	228.16	228.16
Dec 1999	234.63	2.83	8.16	1.08	216.92	445.08
Dec 2000	241.28	2.83	8.16	1.17	206.24	651.33
Dec 2001	248.12	2.83	8.16	1.27	196.09	847.41
Dec 2002	255.15	2.83	8.16	1.37	186.43	1 033.84
...	...	...	...	...	...	...
...	...	...	...	...	...	...
Dec 2019	410.38	2.83	8.16	5.19	79.01	3 107.74
Dec 2020	422.01	2.83	8.16	5.62	75.12	3 182.85
Dec 2021	433.98	2.83	8.16	6.08	71.42	3 254.27
Dec 2022	446.28	2.83	8.16	6.57	67.90	3 322.18
Dec 2023	458.93	2.83	8.16	7.11	64.56	3 386.73

Source MSCI, IEIF calculation

I find that P_0 is now 3.386.73 compared with 3.857.36 previously, a difference of more than 10%...



The discounted income model is very sensitive to the actual sequence of income and not just to its arithmetic or geometric mean growth rate.

As a result, the Gordon-Shapiro model is problematic because its simplifying assumption of constant revenue growth is neither realistic nor neutral.

These chronicles are linked to my activity at the IEIF. a Paris based think tank on real estate where I conduct research into the modelling of major property variables.
For those less familiar with property analysis. these chronicles can be a source of information and a knowledge base. For experts in the field. their purpose is to launch discussions and exchanges on the various subjects I cover.
Some of the chronicles will be based on known and familiar elements. while others will deal with research elements and present some of the results of my work.